BOTT PERIODICITY FOR GROUP RINGS
AN APPENDIX TO
"PERIODICITY OF HERMITIAN K-GROUPS"

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Abstract. We show that the groups $K_n(RG;\mathbb{Z}/m)$ are Bott-periodic for $n\geq 1$ whenever G is a finite group, m is prime to $|G|$, R is a ring of S-integers in a number field and $1/m\in R$.

For any positive integer m there is a Bott element $b_K\in K_p(\mathbb{Z}[1/m];\mathbb{Z}/m)$, where the period $p=p(m)$ is: $2(\ell-1)\ell^{\nu-1}$ if $m=\ell^\nu$ and ℓ is an odd prime; $\max\{8,2^{\nu-1}\}$ if $m=2^\nu$; and $\prod p(m_i)$ if $m=\prod m_i$ is the factorization of m into primary factors.

In this appendix we consider a finite group G of order prime to m, and consider the Bott periodicity map $x\mapsto x\cdot b_K$ from $K_n(R[G];\mathbb{Z}/m)$ to $K_{n+p}(R[G];\mathbb{Z}/m)$ for rings of integers R in local and global fields.

Theorem 0.1. *Assume that m is relatively prime to $|G|$, and that R is a ring of S-integers in a number field F with $1/m\in R$. Then the Bott periodicity maps $b_K: K_n(R[G];\mathbb{Z}/m)\to K_{n+p}(R[G];\mathbb{Z}/m)$ are isomorphisms for all $n\geq 1$.*

Theorem 0.2. *Assume that m is relatively prime to $|G|$, and that R is the ring of integers in a local field F with $1/m\in R$. Then the Bott periodicity maps b_K : $K_n(R[G];\mathbb{Z}/m)\to K_{n+p}(R[G];\mathbb{Z}/m)$ are isomorphisms for all $n\geq 0$,*

Theorems 0.1 and 0.2 are used in [BKO] to show that $KQ_n(R[G];\mathbb{Z}/m)$ also satisfies Bott periodicity. Since this is immediate if m is odd, when the non-Witt part of $KQ_n(A;\mathbb{Z}/m)$ is a summand of $K_n(A;\mathbb{Z}/m)$, this result is primarily interesting for m even and G a (solvable) group of odd order.

Remark. The proofs show that we may replace $R[G]$ by any order in $F[G]$.

The oldest result of this kind is due to Browder, who proved in [1, 2.6] that the Bott periodicity map b_K is an isomorphism for finite fields and $n\geq 0$ (when m is prime, which implies periodicity for all m).

Almost as old is the following folklore result, which includes finite group rings.

Lemma 0.3. *If B is a finite ring and $1/m\in B$, the the Bott periodicity map $K_n(B,\mathbb{Z}/m)\to K_{n+p}(B,\mathbb{Z}/m)$ is an isomorphism for all $n\geq 0$.*

Proof. If $\mathfrak{m}$ is the nilradical of B, then $B_{\text{red}}=B/\mathfrak{m}$ is semisimple. As such it is a product of matrix rings over finite fields. Now $K_*(B;\mathbb{Z}/m)\cong K_*(B_{\text{red}};\mathbb{Z}/m)$ by [9, 1.4]. By Morita invariance, we are reduced to the Browder's theorem that the Bott periodicity map is an isomorphism for finite fields. □

Remark 0.4. *The finite groups $K_n(\mathbb{F}_2[C_2];\mathbb{Z}/8)$ were computed by Hesselholt and Madsen in* [4]*; they are not Bott periodic as their order goes to infinity with n.*

The next step is to consider the semisimple group ring $F[G]$ when F is a number field. We will use the fact that multiplication by b_K is an isomorphism on

$K_n(F;\mathbb{Z}/m)$ for all $n \geq 1$. This is a consequence of the Milnor-Bloch-Kato conjecture (see [6] [12] [11]), and has been observed by several people.

Lemma 0.5. *If F is a number field and m is prime to $|G|$, the Bott periodicity map $b_K : K_n(F[G];\mathbb{Z}/m) \to K_{n+p}(F[G];\mathbb{Z}/m)$ is an isomorphism for all $n \geq 1$.*

Proof. By Maschke's Theorem, $F[G]$ is the product of simple rings $A_i = M_{n_i}(D_i)$. The *Schur index* of the division algebra D_i is the integer r_i such that D_i has dimension r_i^2 over its center F_i. By [2, 27.11], each $n_i r_i$ divides $|G|$ and so is prime to m. Now the composite of the reduced norm $N_{red} : K_*(D_i;\mathbb{Z}/m) \to K_*(F_i;\mathbb{Z}/m)$ and inclusion $\iota^* : K_*(F_i;\mathbb{Z}/m) \to K_*(D_i;\mathbb{Z}/m)$ is multiplication by r_i in either direction, so these are isomorphisms. Since the Bott periodicity map commutes with ι^*, and is an isomorphism on $K_n(F_i;\mathbb{Z}/m)$ for $n \geq 1$, it is an isomorphism on each factor $K_n(A_i;\mathbb{Z}/m)$ in this range. $\square$

Lemma 0.6. *Let Λ be a maximal R-order in a simple factor A of $F[G]$. Then the periodicity map $b_K : K_n(\Lambda;\mathbb{Z}/m) \to K_{n+p}(\Lambda;\mathbb{Z}/m)$ is an isomorphism for $n \geq 1$.*

Proof. Let R_E be the integral closure of R in the center E of A. For each prime ideal $\mathfrak{p}$ of R_E the ring $\Lambda/\mathfrak{p}\Lambda$ is a finite ring. Since the Bott element commutes with the localization sequence for $\Lambda \to A$ we have a commutative diagram (with finite coefficients omitted):

$$\begin{array}{ccccccccc}
K_{n+1}(A) & \to & \oplus K_n(\Lambda/\mathfrak{p}\Lambda) & \to & K_n(\Lambda) & \to & K_n(A) & \to & \oplus K_{n-1}(\Lambda/\mathfrak{p}\Lambda) \\
\simeq\downarrow b_K & & \simeq\downarrow b_K & & \downarrow b_K & & \simeq\downarrow b_K & & \simeq\downarrow b_K \\
K_{n+p+1}(A) & \to & \oplus K_{n+p}(\Lambda/\mathfrak{p}\Lambda) & \to & K_{n+p}(\Lambda) & \to & K_{n+p}(A) & \to & \oplus K_{n+p-1}(\Lambda/\mathfrak{p}\Lambda).
\end{array}$$

By Lemmas 0.5 and 0.3, the four outer verticals are isomorphisms for $n \geq 1$. By the 5-lemma, the middle vertical map is an isomorphism for $n \geq 1$. $\square$

Proof of Theorem 0.1. As noted in [8, 6.5], the groups $NK_*(R[G];\mathbb{Z}/m)$ are zero because $m \nmid |G|$. Therefore the spectral sequence

$$E^1_{p,q} = N^pK_q(R[G];\mathbb{Z}/m) \Rightarrow KH_{p+q}(R[G];\mathbb{Z}/m)$$

(cf. [10, 1.6]) degenerates to yield

$$K_*(R[G];\mathbb{Z}/m) \cong KH_*(R[G];\mathbb{Z}/m).$$

Now $R[G]$ is an order in the semisimple algebra $F[G]$, and is contained in a maximal order Λ. Note that Λ decomposes as a product of maximal orders Λ_i in the simple factors A_i of $F[G]$. Let I be the conductor ideal from the maximal order Λ to $R[G]$; the quotient rings $R[G]/I$ and Λ/I are finite. Since the Bott element $b_K \in K_P(\mathbb{Z}[1/m];\mathbb{Z}/m)$ is compatible with the Mayer-Vietoris sequences [10, 2.2], we have a commutative diagram, where B denotes $\Lambda \times (R[G]/I)$ and C denotes Λ/I:

$$\begin{array}{ccccccccc}
KH_{n+1}(B) & \to & KH_{n+1}(C) & \to & KH_n(R[G]) & \to & KH_n(B) & \to & KH_n(C) \\
\simeq\downarrow b_K & & \simeq\downarrow b_K & & \downarrow b_K & & \simeq\downarrow b_K & & \simeq\downarrow b_K \\
KH_{n+p+1}(B) & \to & KH_{n+p+1}(C) & \to & KH_{n+p}(R[G]) & \to & KH_{n+p}(B) & \to & KH_{n+p}(C)
\end{array}$$

When $n \geq 1$, the outer periodicity maps b_K are isomorphisms by Lemmas 0.6 and 0.3. Hence the middle map b_K is also an isomorphism for $n \geq 1$. $\square$

We now consider the case of local fields.

Theorem 0.7. *Let R be the ring of integers in a local field F with $1/m \in R$. If Λ is a maximal order in a finite-dimensional central simple F-algebra A, then the periodicity maps $K_n(\Lambda;\mathbb{Z}/m) \to K_{n+p}(\Lambda;\mathbb{Z}/m)$, resp., $K_n(A;\mathbb{Z}/m) \to K_{n+p}(A;\mathbb{Z}/m)$ are isomorphisms for all $n \geq 0$, resp., for all $n \geq 1$.*

Proof. There is a division algebra D so that $A \cong M_r(D)$ and an inner automorphism of A identifying Λ with $M_r(\Delta)$, where Δ is the unique maximal order in D; see [2, 26.23]. Let B be the finite residue matrix algebra $M_r(\bar{\Delta})$ of A. By Suslin-Yufryakov [7], $K_n(\Lambda;\mathbb{Z}/m) \cong K_n(B;\mathbb{Z}/m)$. By Lemma 0.3, the Bott map is an isomorphism on $K_n(\Lambda;\mathbb{Z}/m)$ for all $n \geq 0$. Since the Bott element acts naturally on the localization sequence $K_*(B;\mathbb{Z}/m) \to K_*(\Lambda;\mathbb{Z}/m) \to K_*(A;\mathbb{Z}/m)$, it follows that the $K_n(A;\mathbb{Z}/m)$ are also Bott periodic for $n \geq 1$. $\square$

Proof of Theorem 0.2. The proof of Theorem 0.1 goes through, replacing Lemma 0.6 with Theorem 0.7. $\square$

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